THE RECONSTRUCTION CONJECTURE

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Contents

Abstract. The Reconstruction Conjecture says that graphs on at least three vertices have unique vertex-deleted subgraphs (up to isomorphism). In this report we examine the Reconstruction Conjecture and its variations, including a brief history of the problem. Variations of the Reconstruction Conjecture include other types of graphs, such as digraphs and infinite graphs, both of which we analyse.

1. Introduction

The Reconstruction Conjecture is considered by most to be one of the prominent unsolved problems in graph theory [16], while to others, namely Frank Harary, to be a graphical disease infecting all that it contacts [7]. The Reconstruction Conjecture was discovered by Kelly [10] and his doctoral supervisor Ulam [22]. The problem asks whether graphs are uniquely determined by their *vertex-deleted subgraphs* (subgraphs obtained by deleting a single vertex). Graphs which satisfy this property are called *reconstructible*.

In 1957, Kelly published the first major result which shows trees are reconstructible [10]. Since then, many families of graphs have been shown to be reconstructible; for example, small graphs (those with at least 3 and at most 11 vertices) have been shown to be reconstructible by Brendan McKay [12]. In this report we discuss some of these families which are known to be reconstructible, and those which remain unknown. We also discuss families of graphs which are known to not be reconstructible, and variations of the Reconstruction Conjecture for different classes of graphs (which we define in the following section).

We now provide some definitions which are required to state the Reconstruction Conjecture. A *card* of a graph G is an induced subgraph of G consisting of all but a single vertex of G. Indeed, a card of G is precisely a vertex-deleted subgraph. From now on, we will use the term card when referencing such a subgraph of a graph G. The *deck* of a graph G, which we denote $D(G)$, is the multiset of isomorphism classes of all cards of G. We say a graph property is *recognizable* for a graph G if we can determine the property from its deck $D(G)$. For example, the number of vertices in a graph is recognizable. If G and H are graphs with the same deck, then we say G and H are *hypomorphic*. A graph G is *reconstructible* if for every graph H hypomorphic to G, H is isomorphic to G. Otherwise, we say G is *non-reconstructible*.

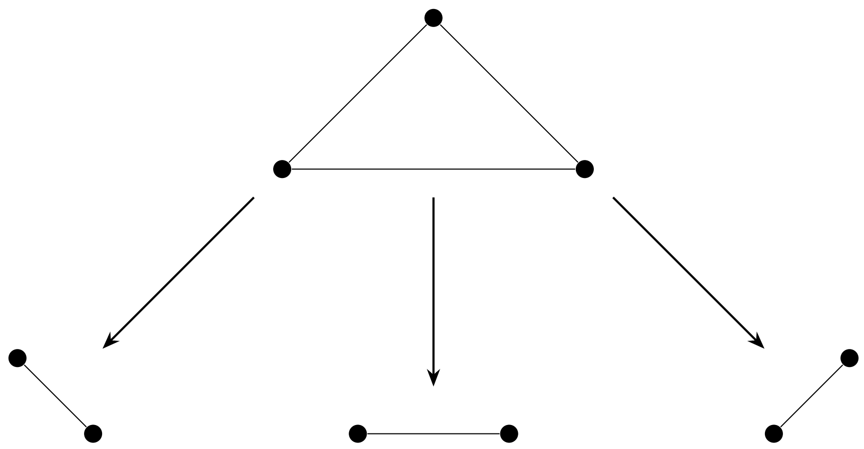

FIGURE 1. Example of obtaining cards

Despite Kelly and Ulam first posing the problem, it is important to note that the current formulation of the Reconstruction Conjecture is by Frank Harary [8] (where we assume graphs to be finite unless otherwise stated).

Conjecture 1.1. (The Reconstruction Conjecture). *Every graph on at least three vertices is reconstructible.*

Example 1.2. Consider the graph G, which is isomorphic to the complete graph on three vertices K_3, shown in Figure 1. We obtain the three cards of G, which are all isomorphic to K_2. It is easy to check that G is the only graph of order 3 with deck $D(G)$, giving us that G is reconstructible.

2. Classes of graphs

In this section we discuss variations of the Reconstruction Conjecture for different classes of graphs.

2.1. Infinite graphs.

We call a graph that is not finite an *infinite graph*, and may simply say that it is *infinite*. Frank Harary proposed the following analogue of the Reconstruction Conjecture for infinite graphs in 1964 [8].

Conjecture 2.1. *Every infinite graph is reconstructible.*

A counterexample to Conjecture 2.1 is provided in the following section.

2.2. Digraphs.

Firstly, we provide some basic definitions to state the variation of the Reconstruction Conjecture for digraphs. A *directed graph*, or *digraph*, is a graph in which edges are given direction. We otherwise say a graph is *undirected*. We may refer to edges of a digraph as *arcs*. If $e = (u, v)$ is an arc, we call u the *initial* vertex and v the *terminal* vertex. The *indegree* of a vertex v, denoted $\operatorname{indeg}(v)$, is the number of arcs with v as the terminal vertex. Similarly, the *outdegree* of v, denoted $\operatorname{outdeg}(v)$, is the number of arcs with v as the initial vertex. A *tournament* is a digraph obtained by giving a direction to each edge in an undirected complete graph.

Frank Harary proposed the following analogue of the Reconstruction Conjecture for digraphs in 1964 [8].

Conjecture 2.2. (Digraph Reconstruction Conjecture). *Every digraph on at least five vertices is reconstructible.*

FIGURE 2. First non-reconstructible example

Counterexamples to Conjecture 2.2 are discussed in the following section. Such counterexamples include both tournaments and non-tournaments of arbitrarily large orders.

This led to another conjecture, posed by Ramachandran and is called the *new digraph reconstruction conjecture* [17]. The new digraph conjecture has neither been proved true nor false. Before we state the new conjecture, let us provide some terminology that is required.

Let D be a digraph with vertices $v_1, v_2, \dots, v_n$, and $D_i = D - v_i$ for each $1 \leq i \leq n$ be the induced subdigraph from deleting the vertex v_i from D. Let $d_i = (\text{outdeg}\,(v_i), \text{indeg}\,(v_i))$, $1 \leq i \leq n$, be the pair of the outdegree of vertex v_i and indegree of vertex v_i, respectively. The collection $\mathscr{D}_D = \{\langle D_i, d_i \rangle \mid 1 \leq i \leq n\}$ is called the *degree pair associated deck*, which we abbreviate to DPA deck. A digraph D is *N-reconstructible* if whenever E is a digraph with $\mathscr{D}_D = \mathscr{D}_E$ (they have the same DPA deck), $D \cong E$ (D and E are isomorphic).

Conjecture 2.3. *Every digraph is N-reconstructible.*

Unlike previous conjectures stated in this section, no counterexample has been found for Conjecture 2.3.

3. Non-reconstructible graphs

Although we do not have a proof or counterexample of Conjecture 1.1 (i.e., a non-reconstructible graph on at least three vertices), there exist non-reconstructible graphs. In this section, we provide a trivial non-reconstructible graph and a non-reconstructible example for infinite graphs. Lastly, we explore the digraph reconstruction conjecture proposed by Harary [8] in 1964.

3.1. First trivial non-reconstructible graph.

Conjecture 1.1 imposes the restriction that graphs must have at least three vertices. This is because of the following example, demonstrating that graphs on two vertices are not uniquely determined by their subgraphs.

Example 3.1. Consider the graphs G and H, shown in Figure 2, which have the same decks. Each card of $D(G)$ and $D(H)$ consists of a single vertex, and hence it is clear G and H are hypomorphic. Furthermore, G and H are not isomorphic, demonstrating G and H are not reconstructible.

3.2. Infinite graphs.

Frank Harary proposed Conjecture 2.1 for infinite graphs [8] in 1964. It was proved by Fisher in 1969 that there exists a countably infinite graph which is not reconstructible [5]. Three years later, in 1972, Fisher, Graham and Harary provided a simpler counterexample [6]. We provide their solution below.

Theorem 3.2. *There exists a countably infinite tree which is not reconstructible.*

Proof. Consider the countably infinite tree T shown in Figure 3. Each vertex in T has infinite degree. Observe that each card of T is a forest consisting of countably infinite copies of T. Now, let F be the forest consisting of two copies of T. Each card of F is a forest consisting of countably infinite copies of T. Hence, it easily follows that T and F are hypomorphic. Thus, there is a countably infinite tree which is not reconstructible. □

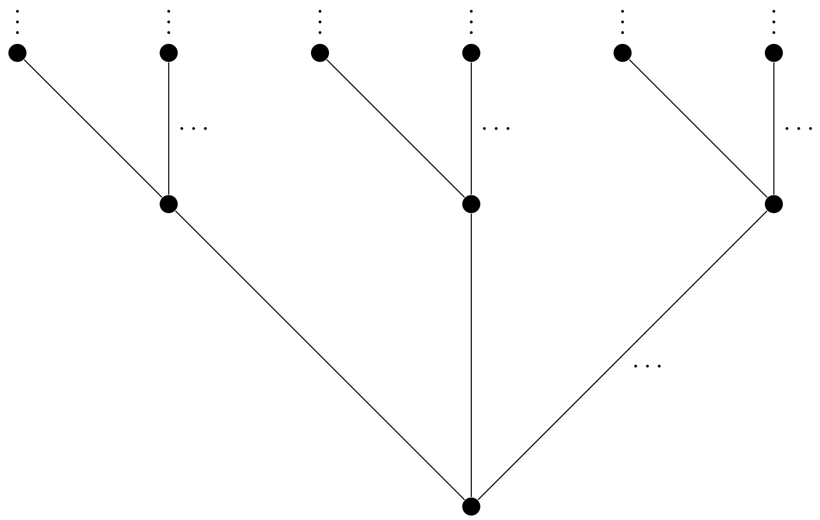

FIGURE 3. Countably infinite tree

Note. A graph is *locally finite* if each vertex has finite degree. Theorem 3.2 prompts the question whether or not locally finite graphs are reconstructible. Harary, Schwenk, and Scott proved that there exists a locally finite forest which is not reconstructible [9]. They then posed the Harary-Schwenk-Scott Conjecture in 1972, which conjectured that locally finite trees are reconstructible. Only very recently, in 2017, was this problem resolved. It was shown by Bowler et al., that locally finite trees need not be reconstructible [4].

3.3. **Digraphs.** Frank Harary proposed Conjecture 2.2 for digraphs on at least five vertices in 1964 [8]. Yet, this was later changed to be on at least seven vertices as non-reconstructible tournaments on both five and six vertices were found by Beineke and Parker in 1970 [2] (shown in Figure 4 and Figure 5, respectively). Despite the conjecture holding for tournaments on 7 vertices, counterexamples were found for tournaments on 8 vertices [17].

Later, in 1977, Stockmeyer proved the digraph reconstruction conjecture to be false by providing counterexamples such as tournaments [19] and non-tournaments [20] of arbitrarily large orders. In particular, in 1977 Stockmeyer gave constructions of non-reconstructible tournaments on 2^t+1 and 2^t+2 vertices (where $t \geq 2$) [19]. Furthermore, Stockmeyer gave constructions of non-reconstructible non-tournament digraphs on $2^s + 2^t$ vertices (where $0 \leq s < t$) [20]. Using Stockmeyer's method, we constructed a pair of non-reconstructible non-tournament digraphs on 5 vertices (taking $s = 0$, $t = 2$), which is seen in Figure 6.

We did this by referring to the *Forming the Digraphs* section in [20], and constructed the adjacency matrices

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix},$$

for the digraphs on the left and right in Figure 6, respectively.

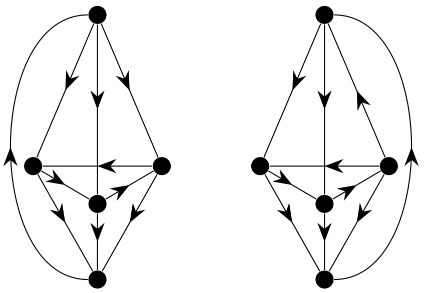

FIGURE 4. (Beineke and Parker, 1970). Non-reconstructible tournaments on five vertices

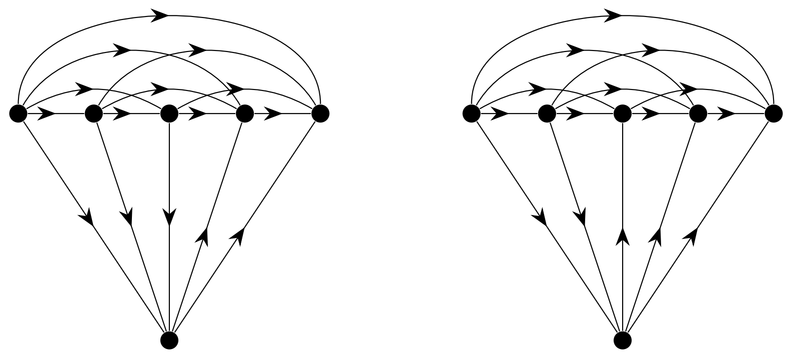

FIGURE 5. (Beineke and Parker, 1970). Non-reconstructible tournaments on six vertices

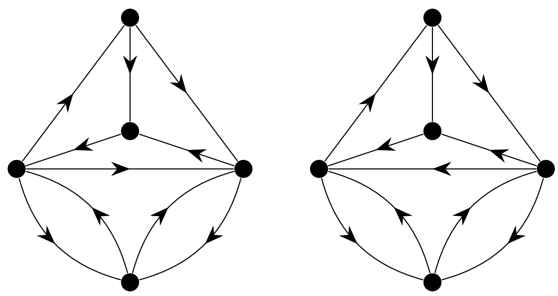

FIGURE 6. Non-reconstructible non-tournament digraphs on five vertices

4. Reconstructible classes

In this section, we will discuss different types of graphs which are reconstructible.

4.1. **Basic results.** Kelly proved that trees, *regular graphs* and disconnected graphs are reconstructible in 1957 [10]. Regular graphs are graphs in which each vertex has the same degree. We provide Kelly's result that regular graphs are reconstructible. To do so, note that the *degree-sequence* in a graph is recognizable. The degree-sequence of a graph is a monotonic nonincreasing sequence of the vertex degrees of its vertices.

Theorem 4.1. *Regular graphs are reconstructible.*

Note. This means that complete graphs K_n and complete bipartite graphs of the form $K_{n,n}$ are reconstructible.

Proof. Suppose G is an n-regular graph. Since the degree-sequence of a graph is recognizable, any graph hypomorphic to G is also n-regular. By taking arbitrary card $C \in \mathscr{D}$, we can obtain an n-regular graph isomorphic to G by adding a single vertex and joining it to the n vertices of C which have degree $n-1$. Thus, G is reconstructible. □

As for other classes of graphs, it should be noted that Ramachandran proved digraphs with a vertex-deleted subgraph isomorphic to a tree are N-reconstructible [17].

4.2. **Small graphs.** Recall that Brendan McKay proved in 1997 that graphs with at least 3 and at most 11 vertices are reconstructible [12]. Yet, it is worth-noting that further results have been obtained; for example, McKay extended this bound to graphs with at most 13 vertices in 2021, and presented the following theorem [13] (which we have modified).

Theorem 4.2. *For at least* 4 *vertices, all graphs in the following classes are reconstructible from their reduced decks (and therefore reconstructible).*

(1) *Graphs with at most* 13 *vertices.*
(2) *Graphs with no triangles, i.e., a* K_3 *subgraph, and at most* 16 *vertices.*
(3) *Graphs with shortest cycle length at least* 5 *and at most* 20 *vertices.*
(4) *Graphs with no cycles of length* 4 *and at most* 19 *vertices.*
(5) *Bipartite graphs with at most* 17 *vertices.*
(6) *Bipartite graphs with shortest cycle length at least* 6 *and at most* 24 *vertices.*
(7) *Graphs with maximum degree at most* 3 *and at most* 22 *vertices.*

It should be noted that the proof required basic group theory and computational effort, testing over 6×10^{13} graphs over 1.5 years [13].

5. Variations

In this section, we consider other formulations of the reconstruction conjecture, of which there are many. For example, one variation uses techniques from analysis, which attempts to reconstruct a graph from metric balls (introduced by Levenshtein, Konstantinova, Konstantinov and Molodtsov) [11].

5.1. **Edge reconstruction.** The *edge reconstruction conjecture* is defined in the natural way; rather than using vertex-deleted subgraphs, we consider *edge-deleted subgraphs* (those obtained by deleting an edge). To formally state the conjecture, we firstly provide some definitions. We will refer to edge-deleted subgraphs as an *edge-card* (see Figure 7). The *edge-deck* of a graph G is the multiset of isomorphism classes of all edge-cards of G. If G and H are graphs with the same edge-deck, then we say G and H are *edge-hypomorphic*. A graph G is *edge-reconstructible* if for every graph H edge-hypomorphic to G, H is isomorphic to G. Otherwise, we say G is *non-edge-reconstructible*.

Frank Harary suggested the following analogue of Conjecture 1.1 in 1964 [8].

Conjecture 5.1. *Every graph with at least four edges is edge-reconstructible.*

Notice that Conjecture 5.1 imposes the restriction that graphs must have at least four edges. This contrasts with Conjecture 1.1, which requires graphs to have at least three vertices (exemplified by Example 3.1). In the following example, we give a graph with three edges which is not edge-reconstructible (but it is reconstructible).

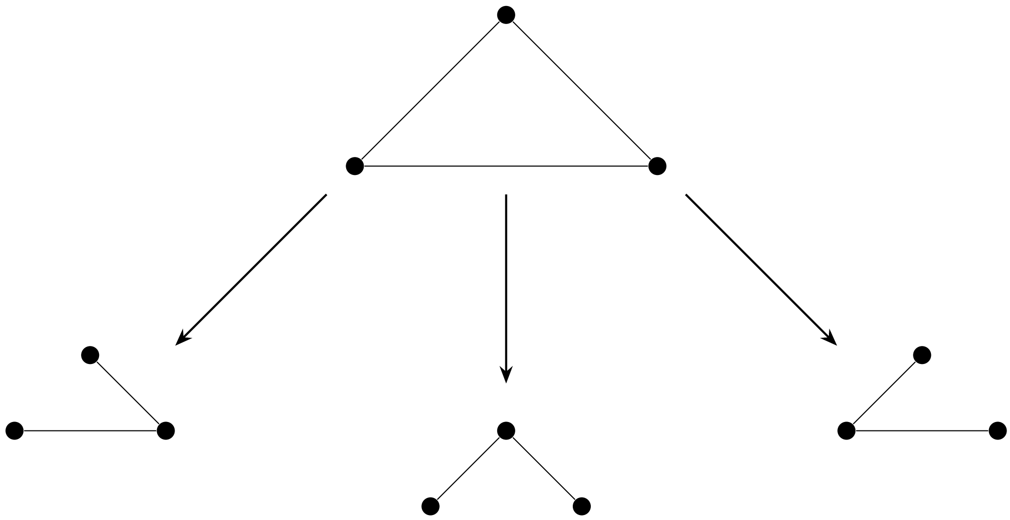

FIGURE 7. Example of obtaining edge-cards

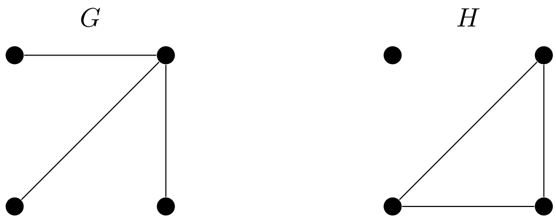

FIGURE 8. Non-edge-reconstructible example

Example 5.2. Consider the graphs G and H, shown in Figure 8. It is easy to check they have the same edge-deck. But they are not isomorphic, and hence they are not edge-reconstructible.

While the edge reconstruction conjecture has not been settled for finite graphs, Thomassen proved in 1977 that there are infinite graphs which are reconstructible but not edge-reconstructible [21]. A simpler (but by no means trivial) example was constructed by Andreae in 1982 [1]. It is natural to ask whether there is an infinite graph which is edge-reconstructible but not reconstructible. We provide a solution to this question below.

Corollary 5.3. *There exists a countably infinite tree which is edge-reconstructible but not reconstructible.*

Proof. Theorem 3.2 gives us that the infinite tree T seen in Figure 3 is non-reconstructible; it is easy to check that it is edge-reconstructible.

Observe that each edge-card of T is a forest consisting of two copies of T. Taking arbitrary edge-card, we can either add an edge joining the two components (resulting in a graph isomorphic to T), or add an edge within one of the components. In the latter case, it is clear that the resulting graph is not edge-hypomorphic to T (simply remove any edge in the component isomorphic to T, which results in an infinite graph consisting of three components). Thus, it follows that T is indeed edge-reconstructible. □

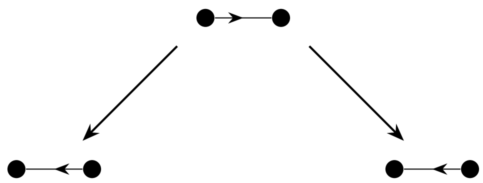

Figure 9. Switching produces graphs isomorphic to original

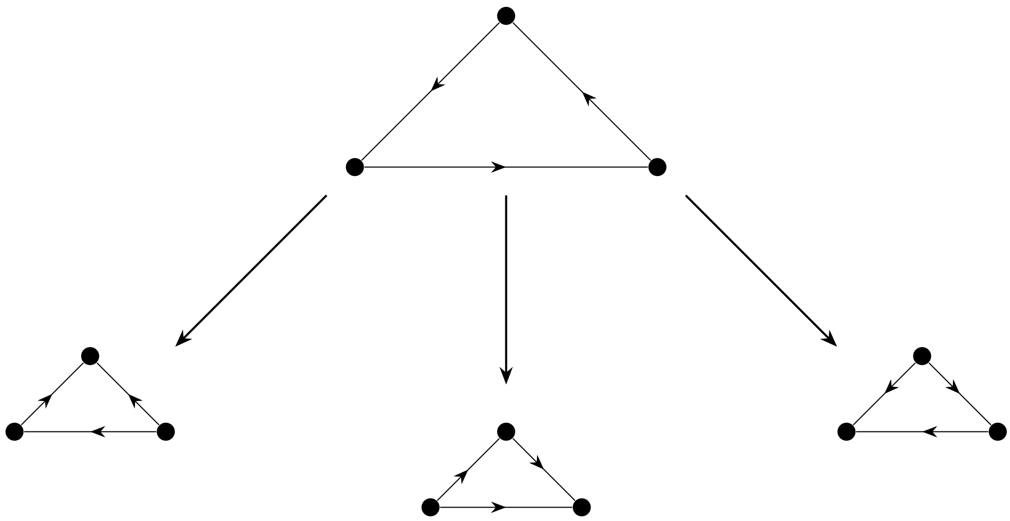

Figure 10. Every switching produces graph non-isomorphic to original

5.2. **Switchings.** In 2011, Bondy and Mercier introduced the *switching* operation on digraphs [3], where a switching at a vertex in a digraph is defined to be the operation which reverses the directions of the arcs incident with that vertex (swap the initial and terminal vertices). It should be noted that Stanley first studied the concept of a switching (which has a different definition to that given by Bondy and Mercier) in 1985 [18].

Bondy and Mercier subsequently asked: for which graphs does every switching produce a graph isomorphic to the original? McKay and Schweitzer solved this problem for *oriented graphs* (digraph with no symmetric pair of arcs) [14]. In Figure 9 we provide an example of an oriented graph with this property. It should be noted, however, that this is not true in general for all classes of graphs (see the tournament in Figure 10, which has every switching produce a graph non-isomorphic to the original).

McLeod, McKay and Faller generalised the problem proposed by Bondy and Mercier, and provided solutions to five similar problems [15]. For example, given an oriented graph we may reverse the direction of a single arc. Another problem is by changing the edge colours (of which there are two) at a vertex of an *edge-coloured graph* (assigning each edge a colour), as seen in Figure 11.

A few open problems were also posed by McLeod et al., including replacing the colours of neighbours in a *vertex-coloured graph* (vertices coloured with 2 colours) [15] (see Figure 12). They did, however, solve the problem when changing the colour of a single vertex in a vertex-coloured graph.

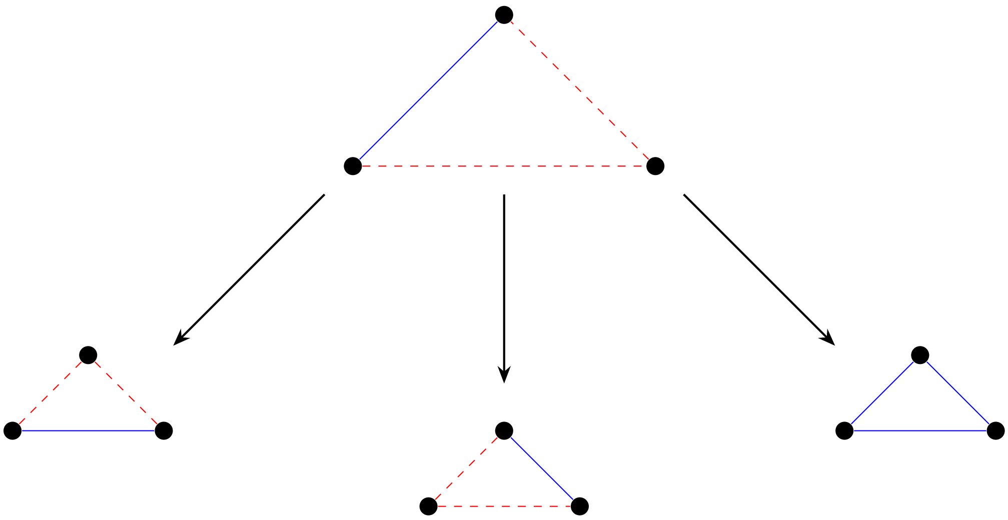

FIGURE 11. Edge colour switching

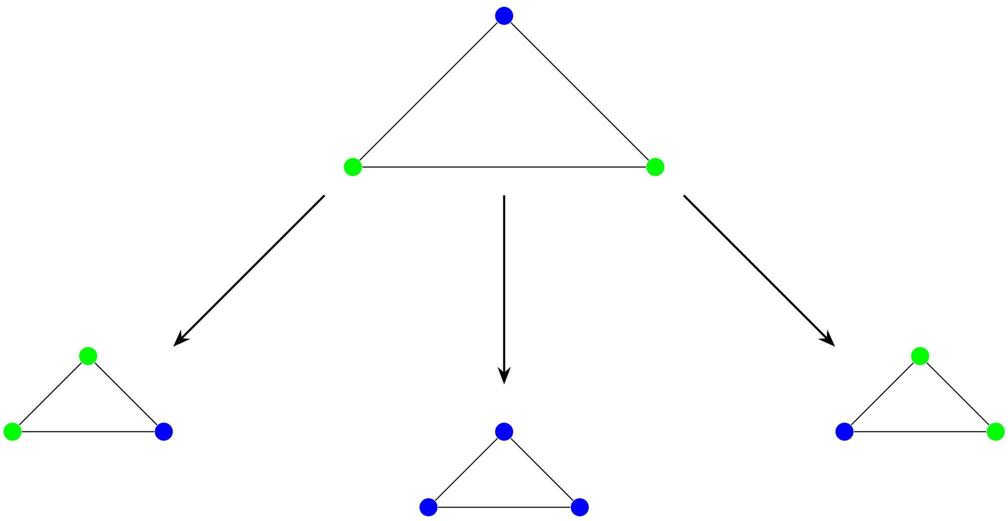

FIGURE 12. Vertex colour switching

6. Conclusion

Many early results primarily use tools from graph theory (and combinatorics). We see that new results, such as those from McKay, use techniques from group theory. It would be interesting in future to see techniques from other branches, such as analysis and topology, used to further our understanding.

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